Greedy Job Selection

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Problem: Given matroid $M = (S, \mathbb{I})$ with wt. $w(s_i)$ on each element $s_i \in S$, find an independent set of max. wt. All wts. are nonnegative.

Algo: Sort elts. so that $w(s_1) \geq w(s_2) \geq \dots \geq w(s_n)$

$T \leftarrow \phi$

for $i = 1 \dots n$

  if $\boxed{T \cup \{s_i\} \in \mathbb{I}}$, $T \leftarrow T \cup \{s_i\}$

Claim: T obtained at the end of the algo is an IS of max. wt.

Proof: Let $T = \{t_1, t_2, t_3, \dots, t_k\}$ in decreasing order of wt.

T_3	t_1			t_2	t_3			t_4		t_5	
s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}

no elt. here extends T_3 → only these elts. can possibly extend T_3

Note: ① Say $X \in \mathbb{I}$, $a \notin X$, $\{a\} \cup X \in \mathbb{I}$.
Then for any $Y \subseteq X$, $\{a\} \cup Y \in \mathbb{I}$

② no elt. before t_i extends T_{i-1} (in example above $i = 4$)

Let $T_i = \{t_1, t_2, t_3, \dots, t_i\}$, i.e., T_i is the first i elts of T.
Actually will show that for all $i \leq k$, T_i is the highest wt. IS of size i.

Base case: clearly true for $i = 0$

Suppose T_{i-1} is the max wt. IS of size $i-1$

For a contradiction, assume $w(\hat{T}_i) > w(T_i)$

By exchange property, $\exists\, a \in \hat{T}_i \setminus T_{i-1}$ s.t. $T_{i-1} \cup \{a\} \in \mathbb{I}$.

But then, a must lie after t_i, or $w(a) \leq w(t_i)$ by Note 2

also, $w(\hat{T}_i \setminus a) \leq w(T_{i-1})$ by Induction hypothesis

Thus $w(\hat{T}_i) \leq w(T_{i-1} \cup t_i)$

$= w(T_i)$ ∎

Do yourself: Now suppose some elements have negative wt.
How do you modify algorithm?
OR: if you want max wt. IS of size k, in presence of $-$ve wts.?
OR: if you want min wt. IS?

Problem: Job Selection

Given: n jobs J

deadlines $d_1 \quad d_2 \quad d_3 \quad \dots \quad d_n$

penalties $w_1 \quad w_2 \quad w_3 \quad \dots \quad w_n$

Each job takes 1 unit of time to finish processing.
We have a single machine

Defn: Given $S \subseteq J$, a schedule is an ordering of the jobs in S

Given $S \subseteq J$ a feasible schedule is an ordering where each job completes on or before its deadline

Given $S \subseteq J$, S is a feasible set if S has a feasible schedule

Problem: Find a feasible set $S \subseteq J$ that minimizes $\sum_{j \notin S} w_j = \sum_{j \in J} w_j - \sum_{j \in S} w_j$

maximize this

Example: 3 jobs $d_1 = 1$, $w_1 = 10$; $d_2 = 1$, $w_2 = 5$; $d_3 = 3$, $w_3 = 7$

Let $D := \max_j d_j$

For any $t \leq D$, given $S \subseteq J$, $N_t(S)$ = # of jobs in S that have deadlines $\leq t$
$= \{j \in S : d_j \leq t\}$

Lemma: Given $S \subseteq J$; the following statements are equivalent:

(i) S has a feasible schedule

(ii) For any $t \leq D$, $N_t(S) \leq t$

(iii) The schedule that orders jobs in S by increasing deadlines is feasible

Proof: (iii) $\Rightarrow$ (i) : trivial

(i) $\Rightarrow$ (ii) : by contradiction

(ii) $\Rightarrow$ (iii) : easy. (do yourself)

Lemma: Let $\mathbb{I} = \{S \subseteq J : S \text{ is feasible}\}$. Then $M = (J, \mathbb{I})$ is a matroid.

Proof: (i) Downward-closed is trivial, if $S \subseteq J$ is feasible then so is any subset

(ii) Let $S, T \subseteq J$ s.t. both are feasible & $|S| > |T|$.

Let $\hat{j} \in S$ be the job w/ the last deadline

Consider $T \cup \hat{j}$.

Say job in T in order of increasing deadline one

$j_1 \quad j_2 \quad j_3 \quad j_4 \quad j_5 \quad j_6 \quad j_7 \quad j_8 \quad \hat{j}$

Claim: The schedule that puts jobs in T in order of increasing deadline & $\hat{j}$ at the end is feasible

Proof: $d_{\hat{j}} \geq |S| > |T|$, hence can schedule $\hat{j}$ after all jobs in T. ∎

(since S is feasible)

Example $S = \{1, 2, 4\}$ (deadlines) $T = \{4, 5\}$ $T \cup \hat{j} = \{4, 5, 4\}$

$\hat{j}$